An overview of fractional attenuation models in exploration geophysics

Yufeng Wang

Stanford Wave Physics Lab

Stanford University

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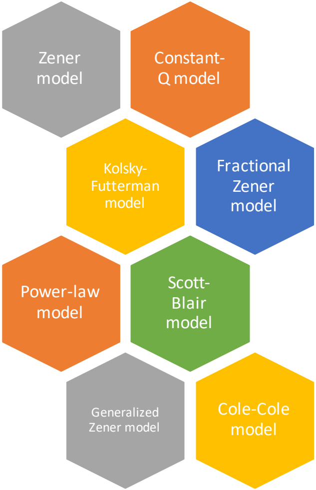
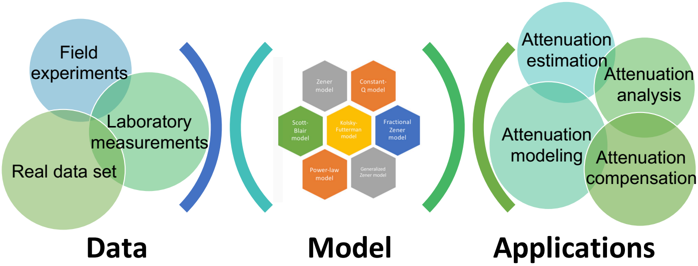
Outline

Have you ever heard of these models?



Why attenuation models matter?



Background

- Wave propagation in subsurface media typically exhibits frequency-dependent absorption and dispersion due to the presence of attenuation mechanism.
- Such an intrinsic attenuation property can be empirically characterized either by experimentally established frequency power law or by physically based mechanical models over a wide range of frequencies.
- Recently, increasing papers present a variety of fractional models that have the ability to describe seismic attenuation with memory property or non-locality.

Motivation

- However, there is absent of profound insight in geophysical community into the connections among these different models from both mathematical and physical viewpoints.
- I firstly revisit the mainstream development of mathematical models of viscoelastic phenomenon in the context of experiments and mechanics, and then we explore how fractional mechanical models lead to a reconciliation of these two apparently different approaches.
- This overview aims at providing geophysicists with more confidence when using fractional models for seismic attenuation modeling, inversion and compensation.

Outline

History of fractional calculus

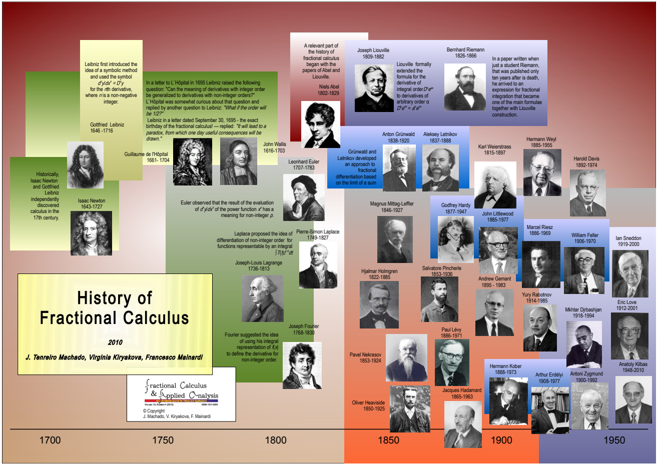


Figure 1. Many great scientists contributed to fractional calculus.

History of fractional calculus

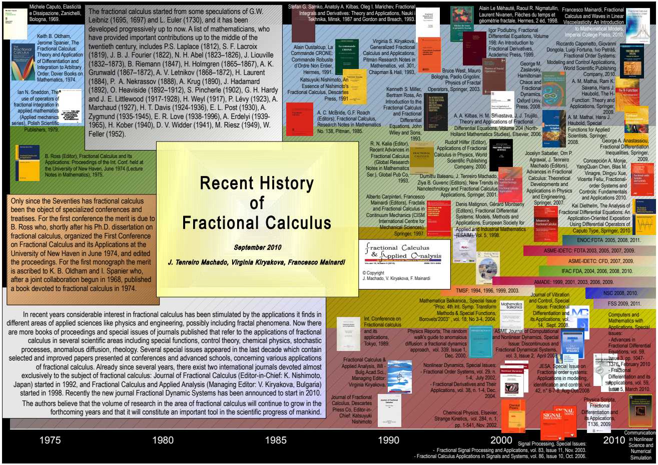


Figure 2. Many great scientists contributed to fractional calculus.

History of fractional calculus

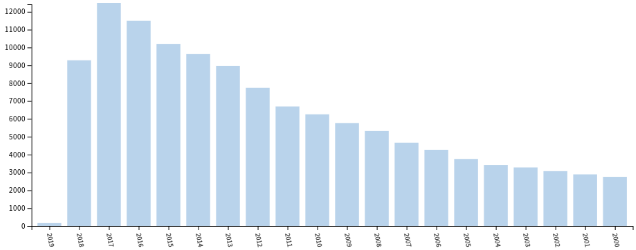


Figure 3. Publication numbers over the past two decades when searching "fractional" on Web of Science.

History of fractional calculus

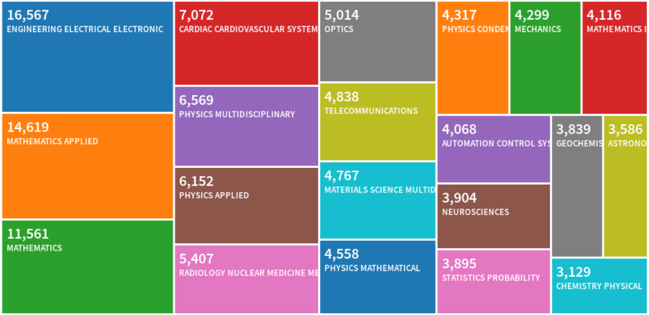


Figure 4. Distribution of subjects over the past two decades when searching "fractional" on Web of Science.

Brief story line

- Fractional calaulus started with a letter between Leibniz and L'Hospital (Sep 30, 1695) on the significance of derivatives of order $1/2$.
- The first great theory of fractional derivation is due to Liouville (1832) and Riemann (1847), which is called Riemann-Liouville integral.
- Grünwald and Letnikov (1867) introduced a definition of fractional derivative based on finite differences.

Brief story line

- Riesz (1936) generalized the Riemann-Liouville integral looking for a solution for some problem in potential theory in connection with partial differential equations for parabolic and hyperbolic cases.
- Caputo (1967) proposed a modified fractional differential operator ${}^{C}D_t^{\alpha}$ in his work on the theory of viscoelasticity.

Riemann-Liouville fractional integral

In our notation, the Cauchy formula reads for $t > 0$:

$$ {}_0I_t^n f(t) := \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \qquad n \in N, \tag{1} $$

where N is the set of positive integers. One can define Riemann-Liouville fractional integral of order $\alpha > 0$:

$$ {}_0I_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad \alpha \in R^+. \tag{2} $$

Riemann-Liouville fractional derivative

Then one can define the Riemann-Liouville fractional derivative of order $\alpha > 0$:

$$ {}_0^{RL}D_t^{\alpha} f(t) := D_t^m \circ {}_0I_t^{m-\alpha} f(t), \qquad m-1 < \alpha < m, \tag{3} $$

then we have

$$ {}_0^{RL}D_t^{\alpha} f(t) := \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}, \qquad m-1 < \alpha < m. \tag{4} $$

Caputo fractional derivative

And the so-called Caputo fractional derivative of order $\alpha > 0$ defined as:

$$ {}_{0}^{C}D_t^{\alpha} f(t) := {}_{0}I_t^{m-\alpha} \circ {}_{0}D_t^{m} f(t), \qquad m-1 < \alpha < m, \tag{5} $$

then we have

$$ {}_{0}^{C}D_t^{\alpha} f(t) := \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}, \qquad m-1 < \alpha < m. \tag{6} $$

Grünwald-Letnikov fractional derivative

In the classical calculus it is well known that derivatives can be expressed as limits of difference quotients,

$$f^{(n)}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t^{n}} \sum_{j=0}^{J} (-1)^j \binom{n}{j} f(t - j\Delta t), \tag{7}$$

this formula can be generalized to fractional case:

$$ {}_{0}^{GL}D_t^{\alpha} f(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t^{\alpha}} \sum_{j=0}^{J} (-1)^j \binom{\alpha}{j} f(t - j\Delta t), \tag{8}$$

where ${}_{0}^{GL}D_t^{\alpha}$ is called the Grünwald-Letnikov fractional derivative operator.

Riesz potential and fractional Laplacian

Riesz introduced n-D integral potential operator with weak singularity

$$(I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n} dy, \tag{9}$$

where the normalized constant $\gamma_n(\alpha)$ is given by

$$\gamma_n(\alpha) = \frac{\pi^n 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}, \; 0 < \alpha < n, \; \alpha - n \neq 2k, \; k \in \mathbb{N}_0. \tag{10}$$

Riesz potential and fractional Laplacian

The definition of fractional Laplacian operator implied by Riesz can be given as follow

$$((-\Delta)^{\alpha} f)(x) = ((-\Delta)^{m}(I^{2m-2\alpha} f)(x). \tag{11}$$

Several researchers look for an operator satisfying the following property:

$$\mathcal{F}((-\Delta)^{\alpha/2} f)(k) = |k|^{\alpha}\mathcal{F}(f)(k), \tag{12}$$

with $0 < \alpha \leq 2$ and

$$|k|^{\alpha} = \left[\sum_{j=1}^{n} k_j^2\right]^{\alpha/2}. \tag{13}$$

Fractional calculus in linear viscoelasticity

Linear viscoelasticity is certainly the field of the most extensive applications of fractional calculus, in view of its ability to model hereditary phenomena with long memory.

- In the first half of 20th century, the early contributors were: Gemant in USA, Scott-Blair in England, Gerasimov and Rabotnov in the former Soviet Union.
- In the late sixties, Caputo and Mainardi explicitly suggested that derivatives of fractional order could be successfully used to model the dissipation in seismology and in metallurgy.

Fractional calculus in linear viscoelasticity

- The beginning of the modern applications of fractional calculus in linear viscoelasticity is generally attributed to the 1979 PhD thesis by Bagley under supervision of Prof. Torvik, followed by a number of relevant papers.
- The 1970 PhD thesis of Rossikhin under the supervision of Prof. Meshkov, and the 1971 PhD thesis of Mainardi under the supervision of Prof. Caputo also contributed greatly to the theory of fractional calculus in linear viscoelasticity.

Outline

Frequency power law

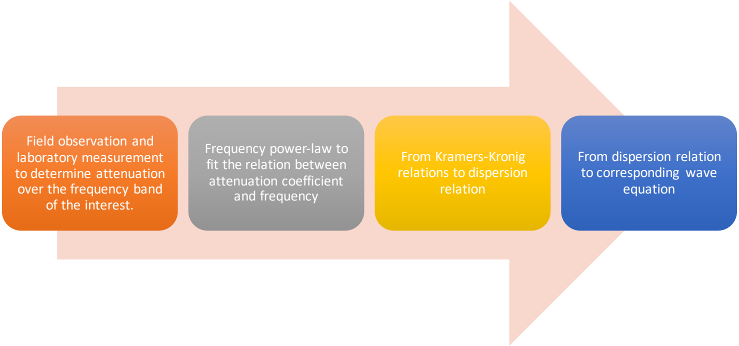


Figure 5. Basic flow of frequency power-law characterization.

Frequency power law

Frequency-dependent attenuation in viscoelastic media typically follows a frequency power law in which the exponent is between 0 and 2 over the frequency range of interest, which can be characterized by an empirical formulation wherein the wavenumber is a complex function of frequency, i.e.,

$$k(\omega) = \frac{\omega}{v_c(\omega)} = \frac{\omega}{v_p(\omega)} - i\alpha(\omega), \tag{14}$$

whose real part and imaginary part separately signifies the phase velocity $v_p(\omega)$ and attenuation coefficient $\alpha(\omega)$.

Frequency power law

If the function of interest can be expressed as a power law or power series, a closed-form solution is available. We assume that the model for the attenuation coefficient is:

$$\alpha(\omega) = \alpha(0) + \alpha_0 |\omega|^y, \tag{15}$$

where α_0 and y are real constants, with $0 < y \leq 2$ typically. Experimental measurements indicate that power-law exponents vary with material types and frequency regimes. For example, the Pierre shale and unconsolidated sediments have nearly linear dependence on frequency (McDonal et al., 1958; Buckingham, 1997).

Frequency power law

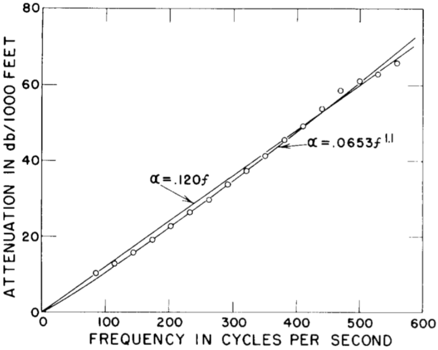


Figure 6. Attenuation of Pierre Shale (McDonal et. al., 1958)[1].

[1] FJ McDonal et al. "Attenuation of shear and compressional waves in Pierre shale". In: *Geophysics* 23.3 (1958), pp. 421–439.

Frequency power law

For power law attenuation of the form given by Eq. 15 with $0 < y < 3$ and $y \neq 1$, the frequency dependence of the sound speed is governed by[2][3]

$$\frac{1}{v_p(\omega)} - \frac{1}{v_p(\omega_0)} = \alpha_0 \tan\left(\frac{\pi y}{2}\right)\left(\omega^{y-1} - \omega_0^{y-1}\right), \tag{16}$$

A general dispersion relation between the spatial wavenumber $k(\omega)$ and the temporal frequency ω that satisfies both Eqs. 15 and 16 can be written as[4]

$$k(\omega) = \frac{\omega}{v_p(\omega_0)} + i\alpha_0 |\omega|^y + \alpha_0 \omega |\omega|^{y-1} \tan\left(\frac{\pi y}{2}\right). \tag{17}$$

[2] Kendall R Waters et al. "On the applicability of Kramers–Krönig relations for ultrasonic attenuation obeying a frequency power law". In: *The Journal of the Acoustical Society of America* 108.2 (2000), pp. 556–563.

[3] Bradley E Treeby. "Acoustic attenuation compensation in photoacoustic tomography using time-variant filtering". In: *Journal of biomedical optics* 18.3 (2013), p. 036008.

[4] James F Kelly, Robert J McGough, and Mark M Meerschaert. "Analytical time-domain Green's functions for power-law media". In: *The Journal of the Acoustical Society of America* 124.5 (2008), pp. 2861–2872.

Kolsky-Futterman model

Here we consider the differential dispersion relations for the case that the exponent y is near an unit $(y \approx n = 1)$, we have

$$\frac{1}{v_p(\omega)} - \frac{1}{v_p(\omega_0)} \approx -\frac{2}{\pi}\alpha_0 \ln\left|\frac{\omega}{\omega_0}\right|. \tag{18}$$

Kolsky and Futterman assumed that frequency-dependent attenuation coefficient $\alpha(\omega)$ is strictly linear with frequency over the range of measurement:

$$\alpha(\omega) = \alpha_0|\omega| = \frac{|\omega|}{2v_p(\omega_0)Q(\omega_0)}. \tag{19}$$

Kolsky-Futterman model

In equation 19, $c_p(\omega_0)$ and $Q(\omega_0)$ are the values of the phase velocity and approximate Q at the reference frequency ω_0. Substituting equation 19 into equation 18, we have

$$\frac{1}{v_p(\omega)} = \frac{1}{v_p(\omega_0)} \left(1 - \frac{1}{\pi Q(\omega_0)} \ln \left| \frac{\omega}{\omega_0} \right| \right) . \tag{20}$$

This is the widely used Kolsky-Futterman model, which can be considered as a special case $(y = 1)$ of frequency power-law model.

Classical mechanical models

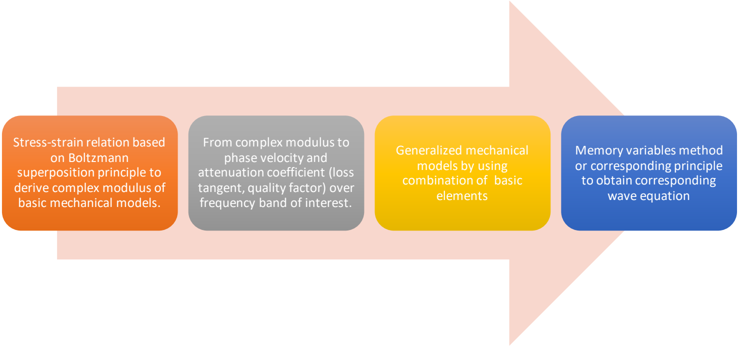


Figure 7. Basic flow of mechanical characterization.

Stress-strain relation

Linear viscoelasticity is founded on the knowledge of two fundamental functions: relaxation function $G(t)$ and creep function $J(t)$. According to Boltzmann superposition principle, the general stress-strain relation can be expressed as

$$\sigma(t) = \int_{-\infty}^{t} G(t-\tau)d\varepsilon(\tau) = G(t) * \frac{d\varepsilon(t)}{dt}, \qquad (21)$$

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)d\sigma(\tau) = J(t) * \frac{d\sigma(t)}{dt}. \qquad (22)$$

The limiting values of the material functions for $t \to 0^+$ and $t \to +\infty$ are related to the glass (G_g, J_g) and equilibrium (G_e, J_e) behaviours of the viscoelastic body, respectively.

Stress-strain relation

In frequency domain, we can define complex modulus and complex compliance as

$$G^*(\omega) = \frac{\hat{\sigma}(\omega)}{\hat{\varepsilon}(\omega)} = i\omega\hat{G}(\omega), \tag{23}$$

$$J^*(\omega) = \frac{\hat{\varepsilon}(\omega)}{\hat{\sigma}(\omega)} = i\omega\hat{J}(\omega), \tag{24}$$

where $\hat{G}(\omega)$ and $\hat{J}(\omega)$ are Fouier transforms of relaxation function $G(t)$ and creep function $J(t)$. We also have

$$G^*(\omega)J^*(\omega) = 1. \tag{25}$$

Loss tangent $\tan\delta$ and quality factor Q

Introducing the phase shift $\delta(\omega)$ between the sinusoidal excitation and sinusoidal response in equations 23 and 24, we can write

$$G^*(\omega) = G^{'}(\omega) + iG^{''}(\omega) = |G^*(\omega)|\mathrm{e}^{+\delta(\omega)}, \tag{26}$$

$$J^*(\omega) = J^{'}(\omega) - iJ^{''}(\omega) = |J^*(\omega)|\mathrm{e}^{-\delta(\omega)}, \tag{27}$$

where $G^{'}(\omega)$ and $J^{'}(\omega)$ are called the storage modulus and storage compliance, while $G^{''}(\omega)$ and $J^{''}(\omega)$ are called the loss modulus and loss compliance.

$$Q^{-1}(\omega) = \tan\delta(\omega) = \frac{G^{''}(\omega)}{G^{'}(\omega)} = \frac{J^{''}(\omega)}{J^{'}(\omega)}, \tag{28}$$

where $\tan\delta(\omega)$ is referred to as the loss tangent and $Q(\omega)$ is quality factor.

Phase velocity v_p and attenuation factor α

Considering waves (k complex, ω real), gives the complex velocity

$$v_c(\omega) = \frac{\omega}{k(\omega)} = \sqrt{\frac{G^*(\omega)}{\rho}}, \tag{29}$$

We define the phase velocity

$$v_p(\omega) = \frac{\omega}{\mathrm{Re}[k(\omega)]} = \left[\mathrm{Re}\left(\sqrt{\frac{\rho}{G^*(\omega)}}\right)\right]^{-1}, \tag{30}$$

and the attenuation factor $\alpha(\omega)$

$$\alpha(\omega) = -\omega \mathrm{Im}\left(\sqrt{\frac{\rho}{G^*(\omega)}}\right). \tag{31}$$

Basic mechanical models

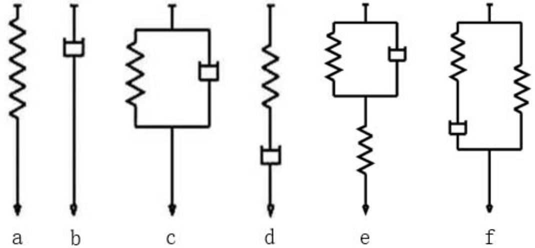


Figure 8. The representations of basic mechanical models including Hooke spring, Newton dashpot, Kelvin-Voigt model, Maxwell model and Zener models.

Zener model

Zener model can be formed by adding a spring either in series to a Kelvin-voigt model or in parallel to a Maxwell model, respectively.

$$\sigma(t) + \tau_\sigma \frac{d\sigma}{dt} = G_e \left(\varepsilon(t) + \tau_\varepsilon \frac{d\varepsilon}{dt} \right), \tag{32}$$

where τ_σ and τ_ε are relaxation time and retardation time, respectively. Typically, we have $0 < \tau_\sigma < \tau_\varepsilon$ and $G_g = G_e(\tau_\varepsilon / \tau_\sigma)$. The corresponding complex modulus is

$$G^*(\omega) = G_e \left(\frac{1 + i\omega\tau_\varepsilon}{1 + i\omega\tau_\sigma} \right). \tag{33}$$

Zener model

The relaxation function $G(t)$ and creep function of Zener model can be obtained by

$$G(t) = \mathcal{F}^{-1}\left[\frac{G^*(\omega)}{i\omega}\right] = G_e\left[1 - \left(1 - \frac{\tau_\varepsilon}{\tau_\sigma}\right)e^{-\frac{t}{\tau_\sigma}}\right]H(t), \tag{34}$$

$$J(t) = \mathcal{F}^{-1}\left[\frac{1}{i\omega G^*(\omega)}\right] = \frac{1}{G_e}\left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\varepsilon}\right)e^{-\frac{t}{\tau_\varepsilon}}\right]H(t). \tag{35}$$

The quality factor $Q(\omega)$ can be determined by

$$Q(\omega) = (\tan\delta(\omega))^{-1} = \frac{G'(\omega)}{G''(\omega)} = \frac{1 + \omega^2\tau_\varepsilon\tau_\sigma}{\omega(\tau_\varepsilon - \tau_\sigma)}. \tag{36}$$

Zener model

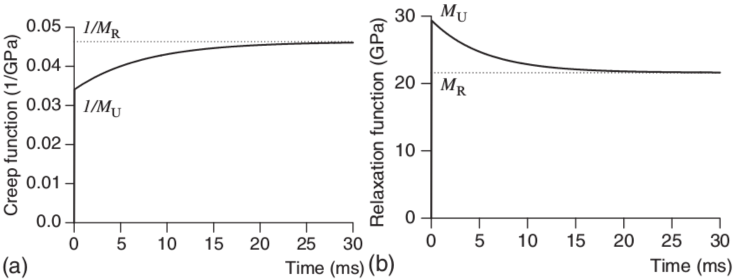


Figure 9. Creep (a) and relaxation (b) functions of the Zener model.

Zener model

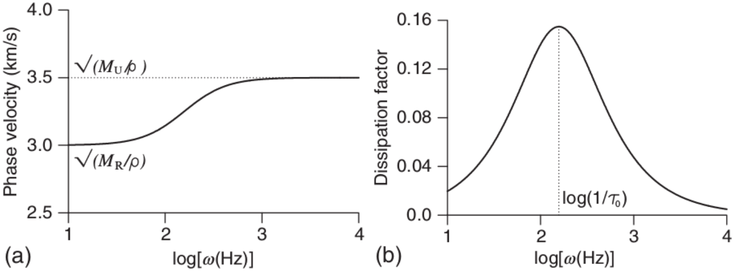


Figure 10. Phase velocity (a) and dissipation factor $Q^{-1}(\omega)$ (b) of the Zener model.

Generalized Zener Model

- The weakness of the classical constitutive relationships is that the macroscopic mechanical behavior of most linear polymers does not exhibit the strong frequency dependence predicted by ordinary derivatives[5]. Consequently, many derivatives (many parameters) are needed in the model alternatively to subtract strong frequency dependence, producing an aggregate weaker frequency dependence.

- Moreover, there is physical evidence that attenuation is almost linear with frequency, therefore Q is constant, in many frequency bands[6].

[5] Ronald L Bagley. "Power law and fractional calculus model of viscoelasticity". In: *AIAA journal* 27.10 (1989), pp. 1412–1417.

[6] McDonal et al., "Attenuation of shear and compressional waves in Pierre shale".

Generalized Zener Model

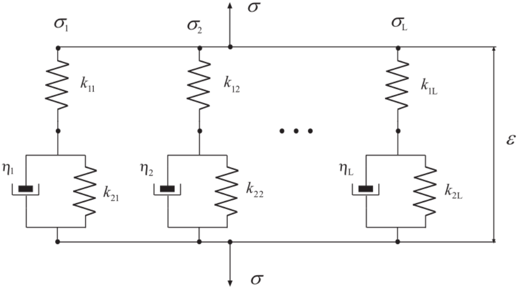


Figure 11. Mechanical model for a generalized Zener material.

Generalized Zener Model

Carcione (2007) consider the parallel system shown in Figure 9, with L Zener elements connected in parallel[7]. The stress-strain relation for each single element is

$$\sigma_l(t) + \tau_{\sigma l}\frac{d\sigma_l}{dt} = G_{el}\left(\varepsilon(t) + \tau_{\varepsilon l}\frac{d\varepsilon}{dt}\right), \qquad l = 1, 2, \cdots, L. \tag{37}$$

According to equation 33, each complex modulus is given by

$$G_l^*(\omega) = G_{el}\left(\frac{1 + i\omega\tau_{\varepsilon l}}{1 + i\omega\tau_{\sigma l}}\right). \tag{38}$$

[7] José M Carcione. *Wave fields in real media: Wave propagation in anisotropic, anelastic, porous and electro-magnetic media*. Vol. 38. Elsevier, 2007.

Generalized Zener Model

The total stress acting on the system in the frequency domain is

$$\hat{\sigma}(\omega) = \sum_{l=1}^{L} G_l^*(\omega)\hat{\varepsilon}(\omega) = \sum_{l=1}^{L} G_{el}\left(\frac{1 + i\omega\tau_{\varepsilon l}}{1 + i\omega\tau_{\sigma l}}\right)\hat{\varepsilon}(\omega). \quad (39)$$

We can choose $G_{el} = G_e/L$, the relaxation function is expressed as

$$G(t) = G_e\left[1 - \frac{1}{L}\sum_{l=1}^{L}\left(1 - \frac{\tau_{\varepsilon l}}{\tau_{\sigma l}}\right)e^{-\frac{t}{\tau_{\sigma l}}}\right]H(t). \quad (40)$$

Generalized Zener Model

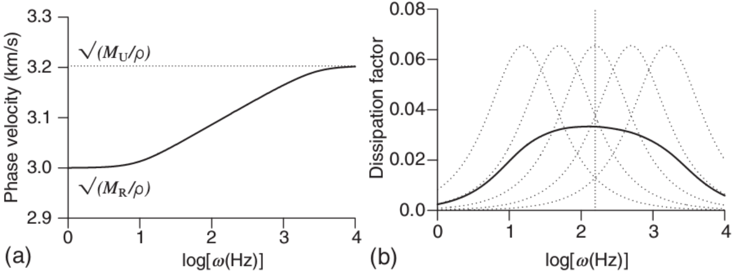


Figure 12. Phase velocity (a) and dissipation factor (b) of the generalized Zener model.

The operator equation

According to the classical theory of viscoelasticity, the general stree-strain relation must be a linear differential equation with constant (postive) coefficients of the following from

$$\left[1+\sum_{k=1}^{p} a_k \frac{d^k}{dt^k}\right] \sigma(t) = \left[m+\sum_{k=1}^{q} b_r \frac{d^k}{dt^k}\right] \varepsilon(t). \qquad (41)$$

Of course, the constants m, a_k and b_k are expected to be subjected to proper restrictions in order to meet the physical requirements of realizability.

Relaxation spectrum and retardation spectrum

By using the conbination rule, general material functions turn out to be of the type

$$G(t) = G_e + \sum_n G_n e^{-t/\tau_{\sigma,n}} + G_{-}\delta(t), \tag{42}$$

$$J(t) = J_g + \sum_n J_n \left(1 - e^{-t/\tau_{\varepsilon,t}}\right) + J_{+}t. \tag{43}$$

In more general cases, the material functions with continuous distributions turn out to be of the following form

$$G(t) = G_e + a\int_0^{\infty} R_\sigma(\tau) e^{-t/\tau} d\tau + G_{-}\delta(t), \tag{44}$$

$$J(t) = J_g + b\int_0^{\infty} R_\varepsilon(\tau)\left(1 - e^{-t/\tau}\right) d\tau + J_{+}t. \tag{45}$$

Fractional mechanical models

One can use differential equations of fractional order to describe viscoelastic behavior intermediate between purely elastic and purely viscous[8][9].

$$\left[1+\sum_{k=1}^{p} a_k \frac{d^{\alpha_k}}{dt^{\alpha_k}}\right]\sigma(t)=\left[m+\sum_{k=1}^{q} b_r \frac{d^{\beta_k}}{dt^{\beta_k}}\right]\varepsilon(t). \qquad (46)$$

Fractional mechanical models has ability to accurately portray measured properties over decades of frequencies of motion with very few parameters.

[8] Christian Friedrich. "Relaxation functions of rheological constitutive equations with fractional derivatives: thermodynamical constraints". In: *Rheological modelling: thermodynamical and statistical approaches*. Springer, 1991, pp. 321–330.

[9] Nicholas W Tschoegl. *The phenomenological theory of linear viscoelastic behavior: an introduction*. Springer Science & Business Media, 2012.

Power-law creep and the Scott-Blair model

Let us consider the viscoelastic solid with the following creep function,

$$J(t) = \frac{1}{\eta\Gamma(1+\nu)} t^{\nu}, \tag{47}$$

such creep behaviour is found to be of great interest in a number of creep experiments. Then the stree-strain relation can be obtained (we consider causal histoies)

$$\varepsilon(t) = \frac{1}{\eta\Gamma(1+\nu)} \int_0^t (t-\tau)^{\nu-1} \sigma(\tau) d\tau = \frac{1}{\eta} \cdot {}_0I_t^{\nu}[\sigma(t)], \tag{48}$$

$$\sigma(t) = \frac{\eta}{\Gamma(1-\nu)} \int_0^t (t-\tau)^{-\nu} \dot{\varepsilon}(\tau) d\tau = \eta \cdot {}_0^C D_t^{\nu}[\varepsilon(t)]. \tag{49}$$

Power-law creep and the Scott-Blair model

Scott-Blair was the first scientist who proposed such a constitutive equation to characterize a viscoelastic material whose mechanical properties are intermediate between those of a pure elastic (Hooke model) and a pure viscous fluid (Newton model). The complex modulus of this model is

$$G^*(\omega) = \eta \cdot (i\omega)^{\nu}, \tag{50}$$

and its loss tangent is

$$\tan\delta(\omega) = \tan(\nu\pi/2). \tag{51}$$

Kjartansson's constant-Q model

Kjartansson (1979) assumed a material that has a creep function of the form

$$J(t) = \frac{1}{M_0\Gamma(1+2\gamma)} \left(\frac{t}{t_0}\right)^{2\gamma} H(t), \tag{52}$$

and the corresponding relaxation function is

$$G(t) = \frac{M_0}{\Gamma(1-2\gamma)} \left(\frac{t}{t_0}\right)^{-2\gamma} H(t). \tag{53}$$

Then complex modulus is

$$G^*(\omega) = M_0 \left(\frac{i\omega}{\omega_0}\right)^{2\gamma}. \tag{54}$$

Kjartansson's constant-Q model

From equation 54, one can obtain the phase velocity $v_p(\omega)$

$$v_p(\omega) = v_0 \left(\frac{\omega}{\omega_0} \right)^{\gamma}, \tag{55}$$

and attenuation coefficient $\alpha(\omega)$

$$\alpha(\omega) = \tan\left(\frac{\pi\gamma}{2}\right) \operatorname{sgn}(\omega) \frac{\omega}{v_p(\omega)}, \tag{56}$$

quality factor

$$Q = (\tan\delta(\omega))^{-1} = \frac{1}{\tan(\pi\gamma)}. \tag{57}$$

Fractional zener model

The fractional Zener stress-strain constitutive relation can be expressed by:

$$\sigma(t)+\tau_\sigma^\nu\frac{d^\nu\sigma(t)}{\partial t^\beta}=G_e\left[\varepsilon(t)+\tau_\varepsilon^\nu\frac{d^\nu\varepsilon(t)}{dt^\nu}\right], \tag{58}$$

where relaxed modulus $G_e=G_g\tau_\sigma^\nu/\tau_\varepsilon^\nu$. Then, the complex modulus is

$$G^*(\omega)=G_e\left(\frac{1+\tau_\varepsilon^\nu(i\omega)^\nu}{1+\tau_\sigma^\nu(i\omega)^\nu}\right), \tag{59}$$

and loss tangent is

$$\tan\delta(\omega)=\frac{(\tau_\varepsilon^\nu-\tau_\sigma^\nu)\omega^\nu\sin(\nu\pi/2)}{1+\tau_\varepsilon^\nu\tau_\sigma^\nu\omega^{2\nu}+(\tau_\varepsilon^\nu+\tau_\sigma^\nu)\omega^\nu\cos(\nu\pi/2)}. \tag{60}$$

Fractional zener model

The relaxation function $G(t)$ and creep function $J(t)$ of fractional Zener model can be obtained by

$$G(t) = G_e \left[1 - \left(1 - \frac{\tau_\varepsilon^\nu}{\tau_\sigma^\nu}\right) E_\nu[-(t/\tau_\sigma^\nu)^\nu]\right] H(t), \tag{61}$$

$$J(t) = \frac{1}{G_e} \left[1 - \left(1 - \frac{\tau_\sigma^\nu}{\tau_\varepsilon^\nu}\right) E_\nu[-(t/\tau_\varepsilon^\nu)^\nu]\right] H(t), \tag{62}$$

where E_ν denotes the Mittag-Leffler function of order ν.

Outline

Rheological representation of Scott-Blair model

The creep function of Scott-Blair model is

$$\begin{aligned} J(t) &= \frac{1}{\eta\Gamma(1+\nu)} t^{\nu} \\ &= \frac{\nu}{\eta\Gamma(1+\nu)\Gamma(1-\nu)} \int_0^{\infty} \frac{1}{\tau^{1-\nu}} \left(1 - e^{-t/\tau}\right) d\tau \\ &= \int_0^{\infty} R_{\varepsilon}(\tau) \left(1 - e^{-t/\tau}\right) d\tau, \end{aligned} \tag{63}$$

where

$$R_{\varepsilon}(\tau) = \frac{\sin(\pi\nu)}{\pi\eta\tau^{1-\nu}}, \tag{64}$$

is the retardation spectrum of Scott-Blair model.

Rheological representation of Scott-Blair model

The retardation spectrum $R_\varepsilon(\tau)$ can be represented by a continuous superposition of Kelvin-Voigt elements in series.

$$J_N(t) = \frac{\sin(\pi\nu)}{\pi\eta}\ln r \sum_{m=0}^{N-1} \tau_m^{\nu}\left(1 - e^{-t/\tau_m}\right), \tag{65}$$

where $\tau_m = \lambda^{(N-m)/N}\mu^{m/N}$ are geometrically spaced with $r = (\mu/\lambda)^{1/N}$. According to Theorem 1 in Papoulia et al.(2010)[10], we have

$$\lim_{N\to\infty} J_N(t) = J(t). \tag{66}$$

[10] Katerina D Papoulia et al. "Rheological representation of fractional order viscoelastic material models". In: *Rheologica Acta* 49.4 (2010), pp. 381–400.

Rheological representation of Scott-Blair model

In a similar way, the relaxation spectrum of Scott-Blair model

$$R_\sigma(\tau) = \frac{\eta \sin(\pi\nu)}{\pi \tau^{1+\nu}}, \tag{67}$$

which can be represented by a continuous superposition of Maxwell units in parallel.

$$G_N(t) = \frac{\eta \sin(\pi\nu)}{\pi} \ln r \sum_{m=0}^{N-1} \frac{1}{\tau_m^\nu} e^{-t/\tau_m}. \tag{68}$$

According to Theorem 2 in Papoulia et al.(2010), we have

$$\lim_{N\to\infty} G_N(t) = G(t). \tag{69}$$

Rheological representation of FMM

The fractional Maxwell model (FMM) consists of a fractional dashpot with constants η, ν in series with a linear spring of stiffness G_g, its creep function

$$J(t) = J_g(t) + \frac{t^\nu}{\eta\Gamma(1+\nu)}, \tag{70}$$

by Theorem 1, this creep function is approximated by

$$J(t) = J_g(t) + \frac{\sin(\pi\nu)}{\pi\eta}\ln r \sum_{m=0}^{N-1} \tau_m^\nu \left(1 - e^{-t/\tau_m}\right). \tag{71}$$

Rheological representation of FKM

The fractional Kelvin model (FMM) consists of a fractional dashpot with constants η, ν in parallel with a spring of stiffness G_e, its relaxation function

$$G(t) = G_e(t) + \frac{\eta}{\Gamma(1-\nu)t^{\nu}}, \tag{72}$$

by Theorem 2, this creep function is approximated by

$$G(t) = G_e(t) + \frac{\eta \sin(\pi\nu)}{\pi} \ln r \sum_{m=0}^{N-1} \frac{1}{\tau_m^{\nu}} e^{-t/\tau_m}. \tag{73}$$

Rheological representation of FMM

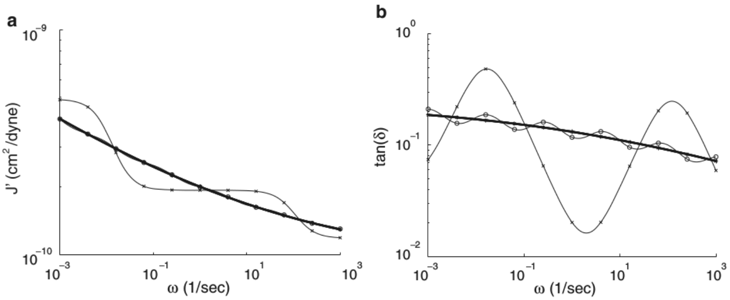


Figure 13. Frational Maxwell model using 3, 9, 27-unit approximation, respectively. (Papoulia et. al., 2010)

Rheological representation of FKM

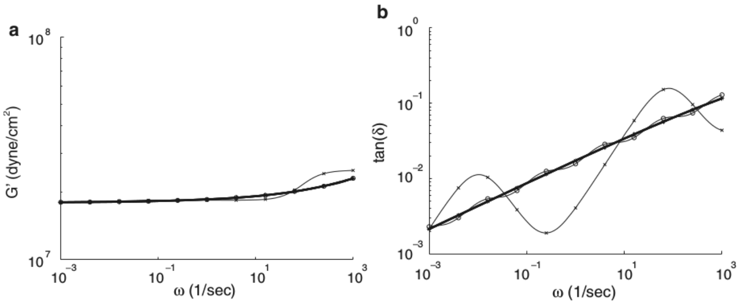


Figure 14. Frational Kelvin-Voigt model using 3, 9, 27-unit approximation, respectively. (Papoulia et. al., 2010)

Rheological representation of FZM

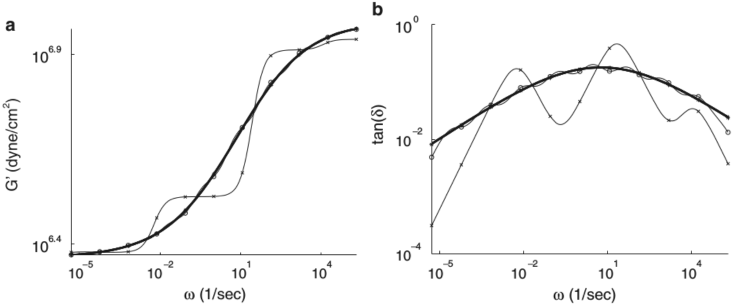


Figure 15. Frational Zener model using 3, 9, 27-unit approximation, respectively.

(Papoulia et. al., 2010)

Physical interpretation

- Frequency power law is a well established model that can fit the experimental measurements over a wide frequency band, which eventually results in the presence of fractional derivative in time domain. Fractional mechanical models also exhibit frequency pawer-law attenuation.
- Fractional mechanical models are the generalization of classical mechanical models by introducing power-law creep functions. The rheological representations converge to the corresponding fractional model in the limit as the number of units tends to infinity.

Physical interpretation

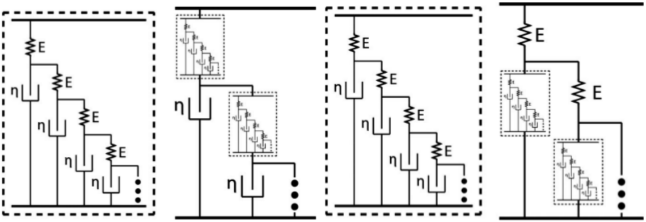


Figure 16. Example of a recursive fractal ladder model, These particular fractal models yield fractional derivative with $\beta = 1/4, 3/4$, respectively. (Kelly, James F, 2009)[11]

[11] James F Kelly and Robert J McGough. "Fractal ladder models and power law wave equations". In: *The Journal of the Acoustical Society of America* 126.4 (2009), pp. 2072–2081.

Physical interpretation

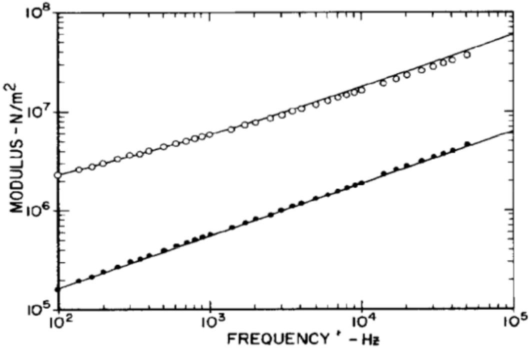


Figure 17. The mechanical viscoelastic properties of polybutadine (with fractional order of 1/2), verified by Rouse's molecular theory[12] (Bagley and Torvik, 1983).

[12] Ronald L Bagley and PJ Torvik. "A theoretical basis for the application of fractional calculus to viscoelasticity". In: *Journal of Rheology* 27.3 (1983), pp. 201–210.

Physical interpretation

Bagley and Torvik (1983): It is not our intention that the empirical model be considered to again verify these theories. Rather, it is our intention to emphasize that the essential features of the empirical model are of the same form as the results of molecular theory. Thus, the empirical model should be viewed as something more than an arbitrary construction which happens to be convenient for the description of experimental data.

Outline

Discussion

- Generally speaking, all of these attenuation models are either based on experimental observation or mathematical (mechanical) approximation, so what is the most fundamental model that can well explain the attenuation? or what is the best criterion to determine which model should be used?

* experimentally fit;
* physically clear;
* mathematically concise;
* computationally efficient;
* . . . .

Discussion

- Fractional model is not merely an empirical generalization of classical model, it may provide more fundamental explanation for dynamic system, in this sense, classical Newton's theory can be considered as a special case of the fractional framework.

* diffusion processes;
* viscoelastic materials;
* polymer physics;
* statistical physics;
* $\cdots$.

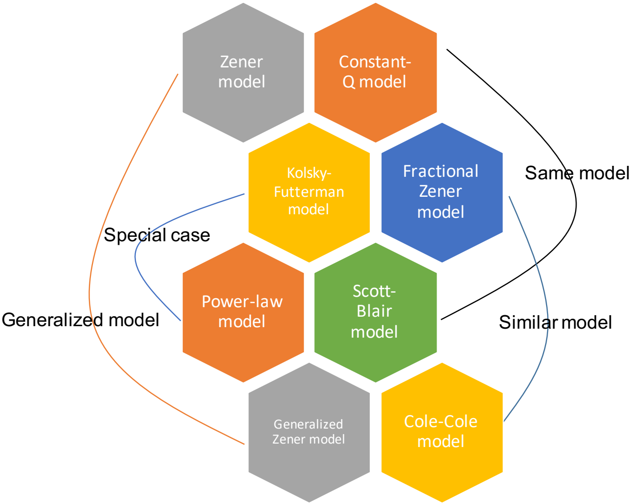
Discussion

- One limitation of fractional models is that the numerical simulation for temporally non-local fractional derivatives occupies large memory due to its requirement to access the history of the wavefields.
- To remedy this drawback, some researchers reformulated the temporal fractional derivatives into the decoupled fractional Laplacians (DFLs) using the smallness approximation and Euler's formula. Then the spatially non-local DFLs can be efficiently computed by the Fourier collocation spectral method.
- How it behaves when this condition is not met?

Conclusions

- Fractional models as a generalization of classical models have the ability to characterize weak frequency-dependent seismic attenuation, which can be represented by classical elements with the number of units tends to infinity.
- Fractional model is not merely an empirical generalization of classical model, it may provide more fundamental explanation for dynamic system. More attention should be paied to this area from physics, engineering and mathematics communities.

They are related



Next talk: applications

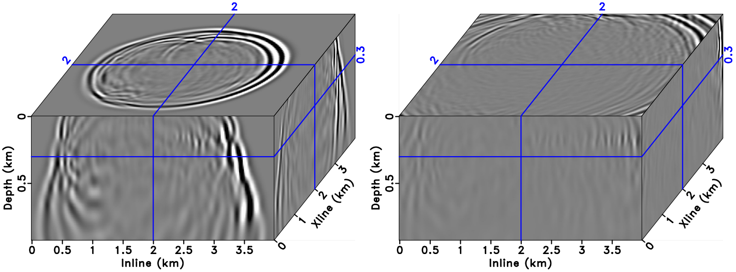


Figure 18. Viscoacoutic modeling using fractional time derivative.[13]

[13] Yufeng Wang et al. "An unsplit convolutional perfectly matched layer for visco-acoustic wave equation with fractional time derivatives". In: *SEG Technical Program Expanded Abstracts 2015*. Society of Exploration Geophysicists, 2015, pp. 3666–3671.

Next talk: applications

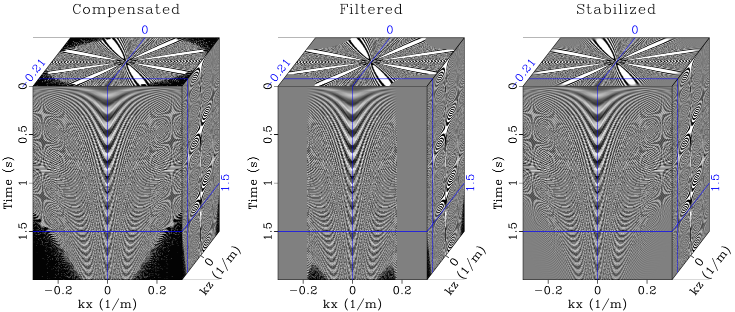


Figure 19. Adaptive stabilization for Q-RTM.[14]

[14] Yufeng Wang et al. "Adaptive stabilization for Q-compensated reverse time migration". In: *Geophysics* 83.1 (2018), S15–S32.

Next talk: applications

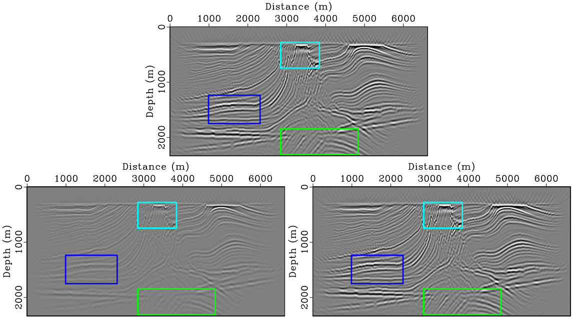


Figure 20. Q compensation for seismic imaging.[15]

[15] Wang et al., "Adaptive stabilization for Q-compensated reverse time migration".

Next talk: applications

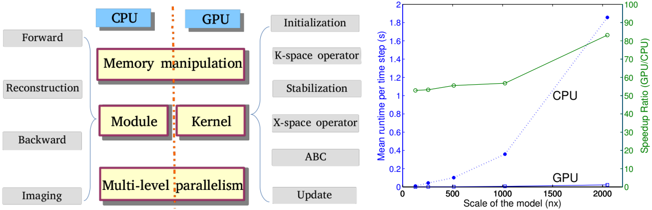


Figure 21. CuQ-RTM: A CUDA-based code package for stable and efficient Q-compensated RTM.[16]

[16] Yufeng Wang et al. "CuQ-RTM: A CUDA-based code package for stable and efficient Q-compensated RTM". In: *Geophysics* 84.1 (2018), pp. 1–69.

Next talk: applications

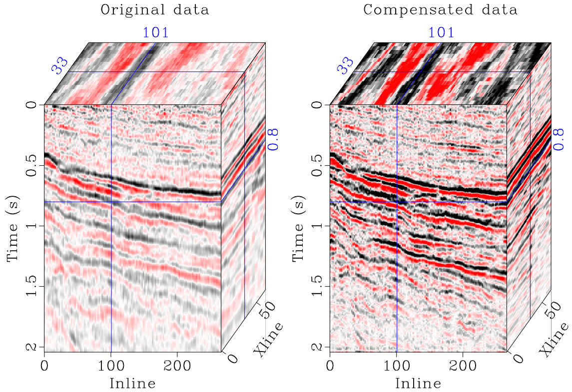


Figure 18. L_{1-2} minimization for exact and stable seismic attenuation compensation.[17]

[17] Yufeng Wang et al. "L1-2 minimization for exact and stable seismic attenuation compensation". In: *Geophysical Journal International* 213.3 (2018), pp. 1629–1646.

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Wang, Yufeng et al. "L1-2 minimization for exact and stable seismic attenuation compensation". In: *Geophysical Journal International* 213.3 (2018), pp. 1629–1646.

Thank you!