Seismic attenuation models: multiple and fractional generalizations

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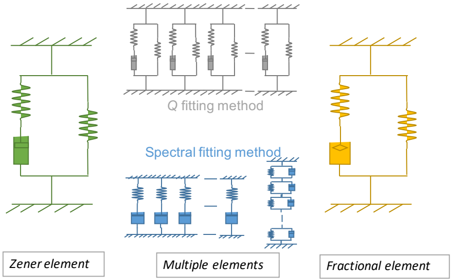
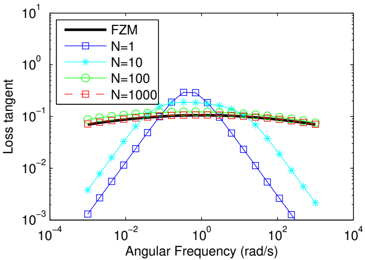
Outline

Motivation


Q fitting method
Spectral fitting method
Zener element
Multiple elements
Fractional element


FZM
N=1
N=10
N=100
N=1000
Loss tangent
Angular Frequency (rad/s)

Motivation

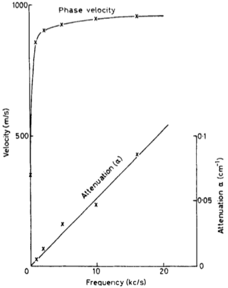


FIG. 1. Experimental measurements of attenuation and velocity in polythene filaments at 10 °C. X, experimental results (Hillier).

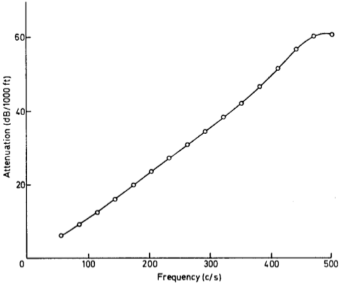


FIG. 2. Attenuation of vertically travelling compressional waves generated by a charge of one pound of dynamite at a depth of 260 ft (Record T). Pierre shale; after McDonal *et al.* (1958).

Motivation

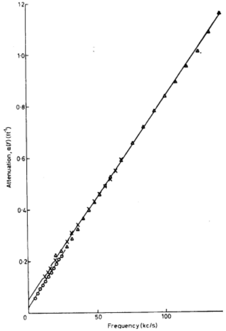


Fig. 4. Wuenschel's experimental values for plate waves in plexiglas sheet (or compressional waves in pseudo-plexiglas) and his least square fit through these points. Different symbols denote that measurements were taken on different occasions.

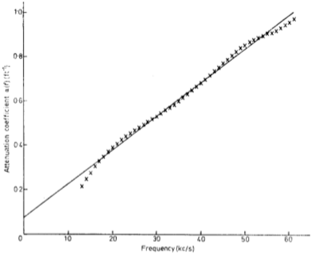


Fig. 5. Attenuation characteristic for shear waves in plexiglas sheet or pseudo-plexiglas after Jordan (1966).

Motivation

- The classical mechanical models such as the Maxwell model, the Kelvin-Voigt model, and the Zener model typically exhibit strong frequency dependence contradictory to many experimental measurements.
- To better capture the viscoelastic behaviour of the subsurface media, multiple and fractional generalizations of the classical models have been intensively studied in the past decades.
- We revisited the mainstream development of mechanical models and investigated the mathematical equivalence between multiple and fractional generalizations by the means of the spectral fitting.

Motivation

- Specifically, we constructed a mechanical model with multiple Maxwell models in parallel to approximate the fractional Zener model. Numerical results show good agreement between these two generalized models.
- This study aims to provide geophysicists with a deep insight into fractional attenuation models and more confidence when using these models for seismic attenuation modeling, inversion and compensation.

Outline

Constitutive law

According to Boltzmann superposition principle, the constitutive equation of linear viscoelasticity can be represented as the convolution integral form:

$$\begin{cases} \sigma(t) = G_g \varepsilon(t) + \int_0^t \dot{G}(t-\tau)\varepsilon(\tau)d\tau, \\ \varepsilon(t) = J_g \sigma(t) + \int_0^t \dot{J}(t-\tau)\sigma(\tau)d\tau. \end{cases} \tag{1}$$

Classical mechanical models

The classical models with simple combination of mechanical elements (no more than three elements) have the following general form of the material functions[1]:

$$\begin{cases} G(t) = G_e + G_1 e^{-t/\tau_\sigma} + G_-\delta(t), \\ J(t) = J_g + J_1\left(1 - e^{-t/\tau_\varepsilon}\right) + J_+ t, \end{cases} \tag{2}$$

where $J_1 = J_e - J_g$ and $G_1 = G_g - G_e$ can be considered as a single constant relaxation (retardation) spectrum.

[1] Francesco Mainardi. *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*. World Scientific, 2010.

Multiple mechanical models

In principle, any time function can be described by a suitable series or parallel ensemble. By using the combination rule, general material functions of multiple mechanical models turn out to be of the type

$$\begin{cases} G(t) = G_e + \sum_{i=1}^{N} G_i e^{-t/\tau_{\sigma,i}} + G_{-}\delta(t), \\ J(t) = J_g + \sum_{i=1}^{N} J_i \left(1 - e^{-t/\tau_{\varepsilon,i}}\right) + J_{+}t. \end{cases} \tag{3}$$

Fractional mechanical models

A genral material functions of fractional models is given by[2]

$$\begin{cases} G(t) = G_e + \sum_{i=1}^{n} G_i E_\nu \left[-(t/\tau_{\sigma,i})^\nu \right] + G_- \frac{t^{-\nu}}{\Gamma(1-\nu)}, \\ J(t) = J_g + \sum_{i=1}^{n} J_i \left[1 - E_\nu \left[-(t/\tau_{\varepsilon,i})^\nu \right] \right] + J_+ \frac{t^{\nu}}{\Gamma(1+\nu)}, \end{cases} \quad (4)$$

where E_ν is the Mittag-Leffler function of order ν. Generally speaking, the fractional mechanical models has ability to accurately portray measured properties over decades of frequencies of motion with very few parameters.

[2] Mainardi, *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*.

Outline

The Zener model

The stree-strain relation of the Zener model can be written as:

$$\sigma(t) + \tau_\sigma \frac{d\sigma}{dt} = m\left(\varepsilon(t) + \tau_\varepsilon \frac{d\varepsilon}{dt}\right). \tag{5}$$

The corresponding complex modulus is

$$G^*(\omega) = m\left(\frac{1 + i\omega\tau_\varepsilon}{1 + i\omega\tau_\sigma}\right), \tag{6}$$

and its loss tangent is given by

$$\tan\delta(\omega) = \frac{G''(\omega)}{G'(\omega)} = \frac{\omega(\tau_\varepsilon - \tau_\sigma)}{1 + \omega^2\tau_\varepsilon\tau_\sigma}, \tag{7}$$

The Zener model

The relaxation function $G(t)$ and creep function $J(t)$ of Zener model can be obtained by

$$\begin{cases} G(t) = G_e + G_1 e^{-t/\tau_\sigma}, & G_1 = \left(\frac{\tau_\varepsilon}{\tau_\sigma} - 1\right) m, \\ J(t) = J_g + J_1 \left(1 - e^{-t/\tau_\varepsilon}\right), & J_1 = \left(1 - \frac{\tau_\sigma}{\tau_\varepsilon}\right) \frac{1}{m}. \end{cases} \quad (8)$$

Fractional Zener model

The fractional Zener stress-strain constitutive relation can be expressed by:

$$\sigma(t) + \tau_\sigma^\nu \frac{d^\nu \sigma(t)}{\partial t^\nu} = m\left[\varepsilon(t) + \tau_\varepsilon^\nu \frac{d^\nu \varepsilon(t)}{dt^\nu}\right]. \tag{9}$$

Then, the complex modulus of this model is represented as

$$G^*(\omega) = m\left(\frac{1 + \tau_\varepsilon^\nu (i\omega)^\nu}{1 + \tau_\sigma^\nu (i\omega)^\nu}\right), \tag{10}$$

and its loss tangent is

$$\tan\delta(\omega) = \frac{(\tau_\varepsilon^\nu - \tau_\sigma^\nu)\omega^\nu \sin(\nu\pi/2)}{1 + \tau_\varepsilon^\nu \tau_\sigma^\nu \omega^{2\nu} + (\tau_\varepsilon^\nu + \tau_\sigma^\nu)\omega^\nu \cos(\nu\pi/2)}. \tag{11}$$

Fractional Zener model

The relaxation function $G(t)$ and creep function $J(t)$ of the fractional Zener model can be obtained by

$$\begin{cases} G(t) = G_e + G_1 E_\nu[-(t/\tau_\sigma)^\nu], & G_1 = \left(\frac{\tau_\varepsilon^\nu}{\tau_\sigma^\nu} - 1\right) m, \\ J(t) = J_g + J_1 \left[1 - E_\nu[-(t/\tau_\varepsilon)^\nu]\right], & J_1 = \left(1 - \frac{\tau_\sigma^\nu}{\tau_\varepsilon^\nu}\right) \frac{1}{m}. \end{cases} \tag{12}$$

Outline

Generalized Maxwell model

A group of Maxwell elements in parallel represents a discrete spectrum of relaxation times, each time $\tau_{\sigma,i}$ being associated with a spectral strength G_i. Since in a parallel arrangement the stresses are additive, it can readily be shown

$$G_{m,N}(t) = \sum_{i=1}^{N} G_i e^{-t/\tau_{\sigma,i}}, \tag{13}$$

where the subscript m denotes multiple Maxwell elements in parallel.

Generalized Kelvin-Voigt model

Similarly, a group of Kelvin-Voigt elements in series represents a discrete spectrum of retardation times, each time $\tau_{\varepsilon,i}$ being associated with a spectral compliance magnitude J_i. Since in a series arrangement the strains are additive, it turns out that for the Kelvin-Voigt model,

$$J_{kv,N}(t) = \sum_{i=1}^{N} J_i \left(1 - e^{-t/\tau_{\varepsilon,i}}\right) , \tag{14}$$

where the subscript kv denotes multiple Kelvin-Voigt elements in series.

Outline

Relaxation and retardation spectral functions

In more general cases, the material functions with continuous distributions turn out to be of the following form

$$\begin{cases} G_\tau(t) = a \int_0^\infty R_\sigma(\tau) e^{-t/\tau} d\tau, \\ J_\tau(t) = b \int_0^\infty R_\varepsilon(\tau) \left(1 - e^{-t/\tau}\right) d\tau. \end{cases} \quad (15)$$

where $R_\sigma(\tau)$ and $R_\varepsilon(\tau)$ are the relaxation and retardation spectral functions of the viscoelastic body.

Frequency-spectral functions

To determine the time-spectral functions from the knowledge of the relaxation and creep functions, Laplace transform theory and Stieltjes transforms are used[3,4]. Introducing the frequency-spectral functions

$$\begin{cases} S_\sigma(\gamma) = a\dfrac{R_\sigma(1/\gamma)}{\gamma^2}, \\ S_\varepsilon(\gamma) = b\dfrac{R_\varepsilon(1/\gamma)}{\gamma^2}, \end{cases} \tag{16}$$

where $\gamma = 1/\tau$ denotes a retardation or relaxation frequency.

[3] Mainardi, *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models.*

[4] Nicholas W Tschoegl. *The phenomenological theory of linear viscoelastic behavior: an introduction.* Springer Science & Business Media, 2012.

The Laplace transform

Differentiating equation 15 with respect to time and applying Laplace transform yields

$$\begin{cases} s\hat{J}_\tau(s) = \int_0^\infty \frac{\gamma S_\varepsilon(\gamma)}{s+\gamma} d\gamma, \\ s\hat{G}_\tau(s) = G_\tau(0^+) - \int_0^\infty \frac{\gamma S_\sigma(\gamma)}{s+\gamma} d\gamma. \end{cases} \tag{17}$$

The Stieltjes transform

Then we introduce the Stieltjes transform function[5]

$$\hat{H}(s) = \int_0^\infty \frac{\gamma S(\gamma)}{s+\gamma} d\gamma, \tag{18}$$

and its inversion may be carried out by Titchmarsh's formula,

$$\gamma S(\gamma) = \frac{1}{\pi} \mathrm{Im} \left[\hat{H}(\gamma e^{-i\pi}) \right]. \tag{19}$$

[5] Bernhard Gross. *Mathematical structure of the theories of viscoelasticity*. Vol. 1190. Hermann, 1953.

From material functions to time-spectral functions

In summary, once we know the Laplace transforms of material functions, the corresponding frequency distributions can be derived by inversion of Stieltjes transforms, and then using equation 16, we can obtain the time-spectral functions.

Time-spectral functions of fractional Zener model

We now consider the corresponding time-spectral functions of the fractional Zener model

$$\begin{cases} G_\tau(t) = G_1 E_\nu[-(t/\tau_\sigma)^\nu] = G_1 \int_0^\infty R_\sigma(\tau) e^{-t/\tau} d\tau, \\ J_\tau(t) = J_1 \left[1 - E_\nu[-(t/\tau_\varepsilon)^\nu]\right] = J_1 \int_0^\infty R_\varepsilon(\tau) \left(1 - e^{-t/\tau}\right) d\tau. \end{cases} \tag{20}$$

with $J_1 = J_e - J_g$ and $G_1 = G_g - G_e$.

Laplace transform and Stieltjes transform

According to the method of Laplace transform and Stieltjes transform, we first apply Laplace transform to $\dot{G}_\tau(t)$

$$s\hat{G}_\tau(s) = G_1 \frac{s^\nu}{s^\nu + (1/\tau_\sigma)^\nu}. \tag{21}$$

According to equations 17 and 18, we have

$$\hat{H}(s) = G_1 \frac{(1/\tau_\sigma)^\nu}{s^\nu + (1/\tau_\sigma)^\nu}, \tag{22}$$

and the inversion of Stieltjes transformation via equation 19 yields

$$\gamma S_\sigma(\gamma) = \frac{1}{\pi} \frac{(\gamma\tau_\sigma)^\nu \sin(\pi\nu)}{1 + (\gamma\tau_\sigma)^{2\nu} + 2(\gamma\tau_\sigma)^\nu \cos(\pi\nu)}. \tag{23}$$

Relaxation spectrum

Substituting equation 23 to equation 16, then we obtain relaxation spectrum of fractional Zener model

$$R_\sigma(\tau) = \frac{1}{\pi\tau} \frac{\sin(\nu\pi)}{(\tau/\tau_\sigma)^\nu + (\tau/\tau_\sigma)^{-\nu} + 2\cos(\nu\pi)}. \qquad (24)$$

Similarly, the retardation spectrum of fractional Zener model has the same form as above.

Relaxation spectrum

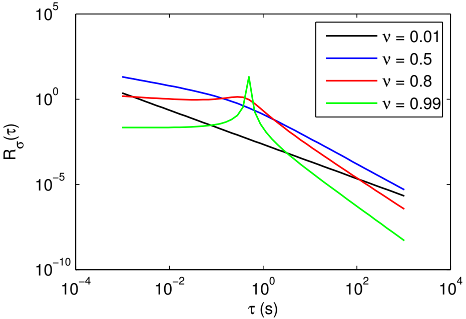


Figure 1. The relaxation spectrum $R_\sigma(\tau)$ of the fractional Zener model with different ν, where $\tau_\sigma^\nu = 0.5$.

Rheological representation

According to Theorem 2 in[6], we can also formulate a rheological representation for fractional Zener model with multiple Maxwell models in parallel. The $G_\tau(t)$ of fractional Zener model is

$$G_\tau(t) = G_1 \int_0^\infty R_\sigma(\tau) e^{-t/\tau} d\tau, \tag{25}$$

And we assume that the $G_{m,N}(t)$ of the generalized Maxwell model is

$$G_{m,N}(t) = G_1 \sum_{i=1}^{N} f_N(\tau_{\sigma,i}) \Delta\tau_{\sigma,i}. \tag{26}$$

[6] Katerina D Papoulia et al. "Rheological representation of fractional order viscoelastic material models". In: *Rheologica Acta* 49.4 (2010), pp. 381–400.

Rheological representation

We assume that the parameters $\tau_{\sigma,i}$ are geometrically spaced between λ and μ $(0 < \lambda < \mu)$ with $\tau_{\sigma,i}/\tau_{\sigma,i-1} = r = (\mu/\lambda)^{1/N}$ to cover a wide band of interest,

$$\tau_{\sigma,i} = \lambda^{(N-i)/N}\mu^{i/N}, \qquad 0 \le i \le N. \tag{27}$$

Lebesgue's dominated convergence

Let $(f_n(x))_{n=1}^{\infty}$ *be a sequence of Lebesgue integrable functions that converge to a limit function* f *almost everywhere on* I*. Suppose that there exists a Lebesgue integrable function* g *such that* $|f_n| \leq g$ *almost everywhere on* I *and for all* $n \in \mathbb{N}$*. Then* f *is Lebesgue integrable on* I *and* $\lim_{n \to \infty} \int_I f_n(x) dx = \int_I f(x) dx$*.*

Rheological representation

According to the Lebesgue's dominated convergence, We enforce

$$\begin{cases} f(\tau) = R_\sigma(\tau)e^{-t/\tau} \\ f_N(\tau_{\sigma,i}) = R_\sigma(\tau_{\sigma,i})e^{-t/\tau_{\sigma,i}}. \end{cases} \tag{28}$$

The $G_{m,N}(t)$ is therefore given by

$$\begin{aligned} G_{m,N}(t) &= G_1 \sum_{i=1}^{N} f_N(\tau_{\sigma,i})\Delta\tau_{\sigma,i} \\ &= G_1(r-1)\sum_{i=1}^{N} \tau_{\sigma,i}R_\sigma(\tau_{\sigma,i})e^{-t/\tau_{\sigma,i}}. \end{aligned} \tag{29}$$

where the forward approximation for numerical integral is used, and $\Delta\tau_{\sigma,i} = \tau_{\sigma,i+1} - \tau_{\sigma,i} = (r-1)\tau_{\sigma,i}$.

Rheological representation

If there exists a Lebesgue integrable function $g(\tau)$ so that $|f(\tau)| \leq g(\tau)$, then we have

$$\begin{aligned}
\lim_{N\to\infty} G_{m,N}(t) &= \lim_{N\to\infty} G_1 \sum_{i=1}^{N} f_N(\tau_{\sigma,i})\Delta\tau_{\sigma,i} \\
&= \lim_{N\to\infty} G_1 \int_0^{\infty} f_N(\tau)d\tau \\
&= G_1 \int_0^{\infty} f(\tau)d\tau \\
&= G_1 \int_0^{\infty} R_\sigma(\tau)e^{-t/\tau}d\tau = G_\tau(t).
\end{aligned} \tag{30}$$

Rheological representation

To this end we need to find a Lebesgue integrable function $g(\tau)$ so that

$$
\begin{aligned}
|f(\tau)| &= R_\sigma(\tau)e^{-t/\tau} \\
&= \frac{1}{\pi\tau}\frac{\sin(\nu\pi)}{(\tau/\tau_\sigma)^\nu + (\tau/\tau_\sigma)^{-\nu} + 2\cos(\nu\pi)}e^{-t/\tau} \\
&\leq \frac{1}{\pi\tau}Me^{-t/\tau} = g(\tau),
\end{aligned}
\tag{31}
$$

where M is the upper bound of the red part.

Rheological representation

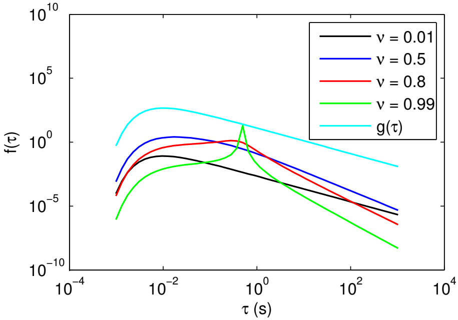


Figure 2. The function $f(\tau)$ with different ν, where $t = 0.01$.

Rheological representation

The complex modulus of the generalized Maxwell model is therefore obtained

$$G^*_{m,N}(\omega) = G_1(r-1)\sum_{i=1}^{N}\tau_{\sigma,i}R_\sigma(\tau_{\sigma,i})\frac{i\omega\tau_{\sigma,i}}{1+i\omega\tau_{\sigma,i}}, \qquad (32)$$

and

$$\lim_{N\to\infty} G^*_{m,N}(\omega) = G^*(\omega), \qquad (33)$$

where $G^*(\omega)$ is the complex modulus of the fractional Zener model shown in equation 10.

The schematic diagrams of the mechanical models

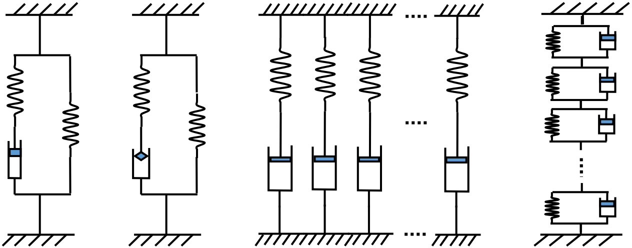


Figure 3. The schematic diagrams of the mechanical models: (a) Zener model, (b) fractional Zener model, (c) generalized Maxwell model, and (d) generalized Kelvin-Voigt model.

Geometrically spaced $\tau_{\sigma,i}$

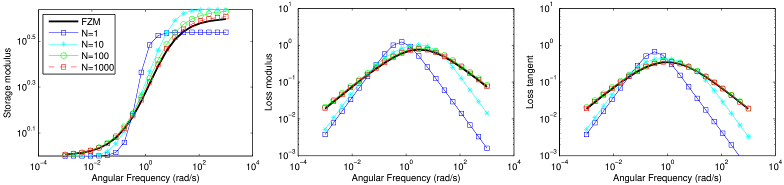


Figure 4. The rheological representations of the fractional Zener model with multiple Maxwell models in parallel: (a) storage modulus, (b) loss modulus, (c) loss tangent. The elements number N = 1, 10, 100, 1000 are respectively shown with different color lines.

Uniformly spaced $\tau_{\sigma,i}$

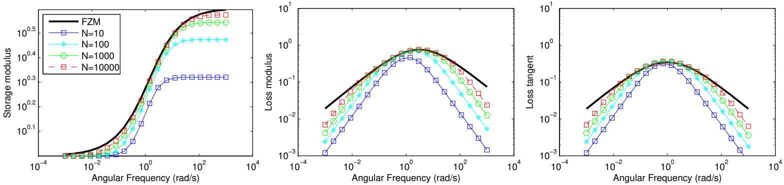


Figure 5. The rheological representations of the fractional Zener model with multiple Maxwell models in parallel: (a) storage modulus, (b) loss modulus, (c) loss tangent. The elements number N = 1, 10, 100, 1000 are respectively shown with different color lines.

Loss tangent versus ν

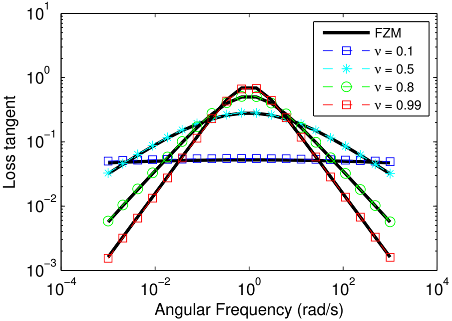


Figure 6. The loss tangent of the fractional Zener model with different ν.

Outline

Summary

- We have developed an explicit mechanical representation with multiple Maxwell elements in parallel for the fractional Zener model. Both mathematical analysis and the numerical comparison demonstrate the equivalence between multiple and fractional generalization of the Zener model.

- Due to the excellent flexibility of the fractional models in fitting the experimental measurements and the increasing popularity of the fractional calculus in viscoelasticity, it is crucial to have a better understanding about the connection between multiple and fractional generalizations from both mathematical and physical viewpoint.

References:

- Mainardi, Francesco. *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*. World Scientific, 2010.
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Thank you!